Midterm Exam — Partial Differential Equations (WBMA008-05)

Wednesday 14 May 2025, 18.30–20.30h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the midterm grade is $G = 1 + p/5$.

Problem 1 (7 + 4 + 4 = 15 points)

Consider the following nonuniform transport equation:

$$\frac{\partial u}{\partial t} + x(x-1)\frac{\partial u}{\partial x} = 0, \quad u(0,x) = \arctan(2x).$$

(a) Show that all nonhorizontal characteristic curves are given by $x(t) = \dfrac{1}{1-ce^t}$ with $c \neq 0$.

(b) Compute the value of the solution u at the point $(t,x) = (1, 1/(1+e))$.

(c) Is the solution u at the point $(t,x) = (1, 1/(1-e))$ determined by the initial condition?

Problem 2 (12 + 3 = 15 points)

Consider the function $f : [-\pi, \pi] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } -\pi \leq x \leq 0, \\ \pi - x & \text{if } 0 < x \leq \pi. \end{cases}$$

(a) Compute the coefficients a_k and b_k of the real Fourier series of f.

(b) What is the value of the Fourier series at $x = 0$?

Problem 3 (10 points)

Consider the following heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial u}{\partial x}(t,0) + u(t,0) = 0, \qquad \frac{\partial u}{\partial x}(t,1) = 0.$$

Show that nontrivial solutions of the form with $u(t,x) = e^{-\omega^2 t} v(x)$ with $\omega \neq 0$ exist if and only if $\tan(\omega) = -1/\omega$.

Turn page for problem 4!

Problem 4 (2 + 3 = 5 points)

Consider the following wave equation with zero initial velocity:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u(0,x) = f(x), \quad \frac{\partial u}{\partial t}(0,x) = 0,$$

where $-\infty < x < \infty$ and $f : \mathbb{R} \to \mathbb{R}$ is at least twice continuously differentiable.

(a) Write down the solution formula of d'Alembert for this problem.

(b) Show that if f is both odd and 2ℓ-periodic, then $u(t,0) = 0$ and $u(t,\ell) = 0$ for all $t \in \mathbb{R}$.

End of test (45 points)

Solution of problem 1 (7 + 4 + 4 = 15 points)

(a) The characteristic curves $t \mapsto (t, x(t))$ are found by solving the following ordinary differential equation:

$$\frac{dx}{dt} = x(x-1).$$

(1 point)

To find the nonhorizontal characteristic curves, we use separation of variables:

$$\begin{aligned} \int \frac{1}{x(x-1)}\,dx = \int dt \quad &\Rightarrow \quad \int \frac{1}{x-1} - \frac{1}{x}\,dx = \int dt \\ &\Rightarrow \quad \log|x-1| - \log|x| = t + k \\ &\Rightarrow \quad \log\left|\frac{x-1}{x}\right| = t + k. \end{aligned}$$

(3 points)

Eliminating the logarithm and absolute value bars gives

$$\frac{x-1}{x} = ce^t \quad \text{where} \quad c = \pm e^k.$$

(2 points)

Finally, solving for x as a function of t gives

$$x = \frac{1}{1 - ce^t}.$$

(1 point)

(b) The point $(t,x) = (1, 1/(1+e))$ lies on the characteristic curve given for $c = -1$.
(1 point)

This characteristic curve intersects the x-axis in the point $(0, \frac{1}{2})$.
(1 point)

Since the points $(1, 1/(1+e))$ and $(0, \frac{1}{2})$ lie on the same characteristic curve and the solution u is constant along such a curve, we have

$$u(1, 1/(1+e)) = u(0, \tfrac{1}{2}) = \arctan(1) = \pi/4.$$

(2 points)

(c) The point $(t,x) = (1, 1/(1-e))$ lies on the characteristic curve given for $c = 1$.
(1 point)

Note that the equation

$$x = \frac{1}{1 - e^t}$$

actually specifies *two distinct curves* in the (t,x)-plane, namely one branch for $t > 0$ and another branch for $t < 0$. The branch that contains the point $(1, 1/(1-e))$ does not intersect the x-axis. Therefore, the solution at the point $(t,x) = (1, 1/(1-e))$ is not determined by the initial condition.
(3 points)

Solution of problem 2 (12 + 3 = 15 points)

(a) For $k = 0$ we obtain the coefficient

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \pi - x\,dx = \frac{1}{\pi}\left[\pi x - \tfrac{1}{2}x^2\right]_0^{\pi} = \tfrac{1}{2}\pi.$$

(2 points)

For $k \geq 1$ the coefficients a_k are given by

$$\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} (\pi - x)\cos(kx)\,dx \\
&= \frac{1}{\pi}\left(\left[\frac{\pi - x}{k}\sin(kx)\right]_0^{\pi} + \frac{1}{k}\int_0^{\pi}\sin(kx)\,dx\right) \\
&= \frac{1}{\pi}\left(\left[\frac{\pi - x}{k}\sin(kx)\right]_0^{\pi} + \frac{1}{k}\left[-\frac{1}{k}\cos(kx)\right]_0^{\pi}\right) \\
&= \frac{1}{\pi k^2}(1 - \cos(k\pi)) \\
&= \frac{1}{\pi k^2}(1 - (-1)^k).
\end{aligned}$$

(5 points)

For $k \geq 1$ the coefficients b_k are given by

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx \\
&= \frac{1}{\pi}\int_0^{\pi} (\pi - x)\sin(kx)\,dx \\
&= \frac{1}{\pi}\left(\left[-\frac{\pi - x}{k}\cos(kx)\right]_0^{\pi} - \frac{1}{k}\int_0^{\pi}\cos(kx)\,dx\right) \\
&= \frac{1}{\pi}\left(\left[-\frac{\pi - x}{k}\cos(kx)\right]_0^{\pi} - \frac{1}{k}\left[\frac{1}{k}\sin(kx)\right]_0^{\pi}\right) \\
&= \frac{1}{k}.
\end{aligned}$$

(5 points)

(b) Extending f to a 2π-periodic function leads to discontinuities at the points $x = k\pi$ with $k \in \mathbb{Z}$. The general theorem about pointwise convergence states that at such points the Fourier series converges to the average of the left and right hand limits. In this case, we obtain

$$\frac{\lim\limits_{x\to 0^-} f(x) + \lim\limits_{x\to 0^+} f(x)}{2} = \frac{0 + \pi}{2} = \frac{\pi}{2}.$$

(3 points)

Solution of problem 3 (10 points)

Substituting the ansatz $u(t,x) = e^{-\omega^2 t} v(x)$ gives the following boundary value problem:

$$v''(x) + \omega^2 v(x) = 0, \quad v'(0) + v(0) = 0, \quad v'(1) = 0.$$

(2 points)

The general solution of the differential equation is

$$v(x) = a\cos(\omega x) + b\sin(\omega x).$$

(2 points)

The boundary condition at $x = 0$ implies that

$$a + b\omega = 0.$$

The boundary condition at $x = 1$ implies that

$$-a\omega\sin(\omega) + b\omega\cos(\omega) = 0.$$

(2 points)

Writing the equations in matrix vector notation gives

$$\begin{pmatrix} 1 & \omega \\ -\omega\sin(\omega) & \omega\cos(\omega) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Nontrivial solutions for a and b exist if and only if the determinant of the coefficient matrix vanishes:

$$\omega\cos(\omega) + \omega^2\sin(\omega) = 0.$$

Since $\omega \neq 0$ this is equivalent to

$$\tan(\omega) = -\frac{1}{\omega}.$$

(4 points)

Solution of problem 4 (2 + 3 = 5 points)

(a) The solution formula of d'Alembert is given by

$$u(t,x) = \frac{f(x+ct)+f(x-ct)}{2}.$$

(2 points)

(b) Since f is odd, we have $f(-z) = -f(z)$ for all $z \in \mathbb{R}$ and thus

$$u(t,0) = \frac{f(ct)+f(-ct)}{2} = \frac{f(ct)-f(ct)}{2} = 0.$$

(1 point)

Again using that f is odd gives

$$u(t,\ell) = \frac{f(\ell+ct)+f(\ell-ct)}{2} = \frac{f(\ell+ct)-f(-\ell+ct)}{2}.$$

(1 point)

Using that f is 2ℓ-periodic gives $f(-\ell+ct) = f(\ell+ct)$ and thus $u(t,\ell) = 0$.
(1 point)